Numerical Boundary Conditions for Viscous Flow Problems

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A method for treating certain troublesome boundary conditions in the numerical solution of time-dependent incompressible viscous flow problems is presented. This method is developed on the basis of an integral representation for the velocity vector which contains the entire kinematics of the problem, including the boundary conditions of concern. It is shown that for the exterior flow problem the freestream condition is satisfied at infinity exactly, and the need to treat a farfield condition is removed by the use of the integral representation. The distribution of a nonvelocity variable on the solid boundary, i.e., the "extraneous" boundary condition needed for both the exterior and the interior flows, are shown to be governed by the kinematics of the problem. The method is shown to accurately follow the local generation of vorticity on the solid boundary computationally.

Introduction

THE development of methods for analyzing general viscous fluid motions, for which the boundary-layer approach is not justifiable, has long remained a central problem for a large number of researchers from various disciplines of engineering, mathematical, and physical sciences. The longevity, intensity, and ubiquity of interest in this subject attested not only to its practical importance in a wide range of application, but also to the formidable difficulties attendant to the analytical solution of the full Navier-Stokes equations. Indeed, the existing analytical solutions of the Navier-Stokes equations are few in number, and are largely restricted to circumstances in which either the equations reduce to linear forms [1], or a similarity variable is found. The contribution of these solutions to the understanding of practical flow problems, while invaluable, are severely restricted in scope. For this reason, recent literature has emphasized the computational approach to the solution of viscous flow problems.

The intense interest in computational methods for viscous flow problems is evidenced by the growth in recent years of research papers on this subject. A review of the literature reveals that progress made in recent years has been extensive, particularly in the application of finite-difference and finite-element techniques. Nevertheless, general viscous flow problems of practical importance remain today mostly beyond the scope of prevailing methods.

The limitation of prevailing methods is known to be especially acute for external flow problems at high Reynolds numbers. This limitation is a direct consequence of two serious obstacles: 1) the excessive computer time and data storage needs for the solution of problems at high Reynolds numbers, and 2) the difficulties and uncertainties associated with the numerical treatment of certain boundary conditions.

A basic cause of the first obstacle is that, for many flows at high Reynolds number, there exists in the flowfield a small region in which the effects of viscous forces are important, and the gradients of field variables are large. This viscous region is embedded in a much larger region of essentially inviscid potential flow in which the gradients are much smaller by comparison. In other words, the length scale for the viscous region is generally much smaller than that for the potential region. Consequently, while closely spaced data points are necessary in the viscous region to provide sufficient resolution and solution accuracy, there is no need for such close spacing in the potential region. In prevailing numerical methods for the Navier-Stokes equations, the potential and viscous regions are solved simultaneously. It then becomes extremely difficult to devise a data grid which provides a sufficient resolution for the viscous region, and yet does not contain an exceedingly large number of data points for the potential region. With increasing Reynolds number, the length scale of the viscous region decreases, and the gradients in the viscous region increase. The total number of data points entering the numerical procedure soon becomes so enormous as to demand a prohibitive amount of computer time and data storage.

Regarding the second obstacle it should be emphasized, even at the risk of appearing trivial, that no meaningful solution of a physical problem involving field variables is possible without an adequate knowledge of the boundary conditions. The Navier-Stokes equations and the continuity equation, whose solutions are being sought, are the governing differential equations common to an astonishingly rich variety of flow patterns. These flow patterns can be drastically different from one another not only quantitatively but also in character, merely because of some differences in the imposed conditions at the flow boundaries. Improper or imprecise numerical treatment of the boundary conditions invariably leads to unreliable or unacceptable solutions. In recent years, a sizable effort has been in progress to analyze the errors in the numerical solution caused by troubles generated at the flow boundaries. Various authors treating different types of flow problems [2-4] are in agreement that the proper handling of boundary conditions are of dominant importance in the numerical solution of flow problems. For the general incompressible viscous flow problem, however, two difficulties relating to the numerical treatment of boundary conditions have remained.

The first difficulty is associated with flows exterior to finite solids. It arises because the flow region is infinite in extent, at least in the mathematical sense. Consequently, boundary conditions imposed at infinity need to be satisfied and, for numerical solution, the infinite region needs to be represented with a finite number of grid points. To resolve this difficulty, some investigators have imposed boundary conditions at a "sufficiently large" distance from the body, rather than at infinity, to limit the flow region. This approach requires a prior knowledge of the solution at large distances. Otherwise, the validity of the solution thus obtained can only be tested by repeating the computations for successively larger regions and comparing the results. This difficulty, referred to in this paper as the farfield boundary condition, has motivated recent developments of transformation techniques for numerically generating body-fitted coordinate systems. [5]

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The second difficulty is associated with both the exterior and the interior flow problems. It arises because the Navier-Stokes equations contain more than the velocity vector as dependent variables while the proper boundary condition for the physical problem is that for the velocity vector. The values of the nonvelocity field variable, e.g., the vorticity, however, must be known in order to initiate, and to proceed with the numerical solution. Since the specification of the values for the nonvelocity variable on the boundary of the fluid domain apparently overspecifies the problem, the expression "extraneous boundary condition" is used in this paper to describe this difficulty. In particular, the difficulty associated with the extraneous vorticity condition on solid boundaries is treated.

In most of the previous investigations, one-sided difference formulae have been used to estimate the vorticity values on solid boundaries from the no-slip condition and the computed velocity (or stream function) values at grid points near the boundary.[2] It has been found that difference formulae of first-order accuracy tend to yield stable solutions, while second-order formulae tend to give unstable results at high Reynolds numbers. In fact, for some problems, second-order formulae, even when stable, gave less accurate solutions than did the first-order formulae.[2] The use of first-order formulae, however, restricts the overall accuracy of the solution. There existed, therefore, considerable uncertainties regarding the correct one-sided formulae to use, and regarding the limitations of each formula.[2]

In several earlier articles, the present author and his co-workers[6-10] discussed a method of solution for the viscous flow problem in which the Navier-Stokes and continuity equations are recast into a system of integro-differential equations. The distinguishing feature of this method is that, for the incompressible viscous flow problem, the solution field can be confined to the viscous region of the flow. The method is therefore especially advantageous to problems where the viscous region occupies only a small part of the total flowfield. The purposes of the present paper are twofold, 1) to present a new method, developed on the basis of the integro-differential formulation, for treating the far-field and the extraneous boundary conditions, and 2) to demonstrate by analyses and numerical illustrations that this method correctly simulates the physical problem and eliminates the previously experienced difficulties associated with boundary conditions.

In Ref. 7, the vector potential, a principal solution in the vector form, and a Green's theorem for vectors, are utilized to derive an integral representation for the vel. vector in incompressible flows. Following this formal procedure, integral representations are also derived for the stream function and vector potential, for the velocity vector in compressible flows,[10] and for the vorticity vector (by recasting the vorticity transport equation) in steady incompressible flows.[10] It should be noted that an integral representation for the velocity vector is obtainable on the basis of the Biot-Savart law without using the formal mathematical procedures. Lighthill,[11] in fact, outlined the basic ideas of an integro-differential approach on the basis of the Biot-Savart Law. Fundamental to Lighthill's concept is the construction of a potential velocity field to insure the satisfaction, on solid boundaries, of the specified conditions for the normal vel. component. Lighthill suggested that this potential velocity field is sometimes described as the field of the image vorticity since, for simple shapes of boundary, it can be related to a distribution of "virtual" or "image" vorticity within the body. Indeed, prior to Lighthill's description of the approach in Ref. 11, Payne treated the two-dimensional problems of a plane jet exiting normally from an opening in an infinite flat plate,[12] and of a flow past a circular cylinder[13] using image vorticity distributions. Mathematically, the method of image is equivalent to the method of Green's function. The difficulties in finding Green's functions for other than simple shapes of boundary no doubt discouraged attempts to generalize Payne's work. Most recently, however, Panniker and Lavan[14] used a conformal transformation technique, together with a Green's function for a circle to treat the flow past an impulsively started elliptical cylinder. Schmall and Kinney[15] recently treated the problem of a lifting flat plate using Lighthill concept. Since the problem considered by them involved a body (flat plate) of infinitesimal thickness, the virtual vorticity in the body appeared as "bound vortex" occupying the location of the plate.

The present approach does not utilize image or virtual vorticity concept. Rather, the "real" instantaneous vorticity in both the fluid and the solid regions is considered. For the special case of a flat plate moving in its own plane,[16] the potential velocity field of Lighthill vanishes because of flow symmetry about the plate surface. For this case, the virtual vorticity in the solid region is zero, and Lighthill's approach becomes identical to the present approach. In general, however, there exist major differences in the implementation of the vorticity boundary condition at solid surfaces and in the determination of the velocity field.

In addition to the above references, recently Chorin[17] discussed a method also based in part on the Biot-Savart integral formulation. In the method of Chorin the values of the velocity field near a boundary are not all computed, but are sampled. The statistical simulation of vortex generation and dispersal in the method makes it possible to analyze flows at high Reynolds numbers; but in so doing the pointwise convergence in either space or time is lost. Bratnow and Ecer[18] also used the Biot-Savart formulation in their finite-element study of the oscillating wing problem. In its computational forms, the method is dissimiliar to that described here.

Integro-Differential Formulation

The Navier-Stokes and continuity equations for the time-dependent flow of a fluid with uniform density ρ and kinematic viscosity ν and subject to negligible body forces are expressible in terms of the velocity vector v and the pressure p as

$$\frac{\partial v}{\partial t} + (v \cdot \Delta)\ v = -\frac{1}{\rho} Rk + \nu \Delta^2 v \tag{1}$$

$$v \cdot v = 0 \tag{2}$$

Equation (1) is valid in the region R occupied by the fluid. If the boundary of R is B, then, with the velocity boundary condition

$$v = v_b(r,t) \text{ on } B \tag{3}$$

where y_b is a known function of the position vector r and the time t, Eqs. (1) and (2) are sufficient for the determination of v and ρ throughout R. It is convenient, however, to introduce the vorticity vector ω defined by

$$\Delta \times v = \omega \tag{4}$$

and partition the problem into its kinematic and kinetic aspects.

The kinetic aspect of the problem describes the process of vorticity diffusion and convection in the fluid domain. In the differential form this kinetic aspect consists of the familiar vorticity transport equation obtained by taking the curl of each term in Eq. (1), and using Eqs. (2) and (4):

$$\frac{\partial \omega}{\partial t} = \nabla x(\nu x \omega) + \nu \nabla^2 \omega \tag{5}$$

With known spatial distributions of v and ω in R at any given time level τ, the new distribution of ω in the fluid domain away from B can be computed for a subsequent time level, say $\tau + \Delta t$, on the basis of Eq. (5). Following Ref. 15, the vorticity distribution in the fluid domain away from B

shall be called the free vorticity. It can be easily seen from Eq. (5) that the time rate of change of ω is zero, wherever ω and its first and second spatial derivatives are zero. Therefore, in the computation of the new vorticity values, it is only necessary to know the values of velocity at the given time τ in the region where ω and its spatial derivatives are non-negligible. Since this region is identical to the viscous region of the flow, the kinetic part of the computation can be confined to the viscous region.

The kinematic aspect of the problem relates the velocity distribution at any given instant to the vorticity distribution at that instant and vice versa. Once the new vorticity distribution in R at $\tau + \Delta t$ is computed (the method of computing the new vorticity distribution on B will be discussed later), the new velocity distribution corresponding to it can be computed using the kinematic relations. This then completes a computation loop to advance the solution by one time step.

If the velocity field is known, then the corresponding vorticity field is determinate on the basis of Eq. (4). On the other hand, the velocity field is uniquely defined by, in addition to Eq. (4), Eq. (2) and the velocity boundary condition (3). In this context, the kinematics of the problem consists of Eqs. (2) and (4) subject to the condition (3).

The usual method of evaluating v from a known distribution of ω is to take the curl of Eq. (4) and, by using Eq. (2), obtain a vector Poisson's equation for v. Since the Poisson's equation is elliptic, the appropriate boundary condition is either a Dirichlet or a Neumann condition (or a combination of the two), specified over a closed boundary. Finite-difference solution of v at each data point in the flowfield is dependent on a knowledge of v value at its neighboring points, and utimately dependent on the boundary condition. It is therefore not possible to isolate a portion of the flowfield and obtain solutions for v in only that portion. As a result, the kinematic computation of v values from the Poisson's equation must cover the entire flowfield, including both the viscous and the potential regions, even though in the kinetic computation of new vorticity values, the only values of v needed are those in the viscous region.

Some investigators have used the stream function-vorticity formulation. Others have developed finite element methods for the problem.[19,20] These methods also must treat the entire flowfield. It can be shown,[7] however, that it is possible to recast the kinematic aspect of the problem as an integral representation for the velocity vector which permits the explicit, point by point, evaluation of v from known distribution of ω. The integral representation is[7]

$$v(r,t) = -\frac{1}{A}\left[\int_R \frac{\omega_0 \times (r_0 - r)}{|r_0 - r|^d} dR_0 - \oint_B \frac{(v_b \cdot n_0)\,(r_0 - r) - (v_b \times n_0) \times (r_0 - r)}{|r_0 - r|^d} dB_0\right] \quad (6)$$

where the subscript '0' indicates that the variables and the integrations are performed in the r_0 space, i.e., $\omega_0 = \omega(r_0,t)$, etc., n_0 is the outward normal unit vector, $A = 4\pi$ and $d = 3$ for three-dimensional problems, $A = 2\pi$ and $d = 2$ for two-dimensional problems. This integral representation constitutes the entire kinematics of the problem. That is, it is completely equivalent to the differential Eqs. (2) and (4) subject to the boundary condition (3).

If ω at time level $\tau + \Delta t$ is established in R, and v_b is specified on B, then the integral representation (6) permits the explicit, point by point, computation of the new velocity distribution at $\tau + \Delta t$. In particular it is now possible to compute the velocity distribution only in the viscous region. As noted earlier, the kinetic computation of new ω values requires a knowledge of v only in the viscous region. Thus the entire solution procedure, including both the kinematic and kinetic parts, can now be confined to the viscous region.

Suppose the viscous region of the flow contains N data points. The numerical quadrature of the first integral in Eq. (6) then requires on the order of N algebraic operations. If this integral is to be evaluated at each of the N points, then the total number of operation is on the order of $N.^2$ The present method therefore appears efficient compared to prevailing methods only if N is very small compared to P, the number of data points needed to cover the entire flowfield. In reality, however, Eq. (6) can be used to compute only the velocity values at data points surrounding the viscous region. Once this is accomplished, a finite-difference method can be utilized for computing velocity values at data points within the viscous region. In this manner, a substantial reduction in computational efforts is realizable, even if N is not very small compared to P.

Farfield Boundary Condition

The contribution to the velocity field by the velocity boundary condition is given by the second integral in Eq. (6). This contribution is in fact a solution of a Laplace's equation for a velocity vector subject to the boundary condition, Eq. (3). This contribution therefore represents a potential flow velocity field, and may be thought of as the flowfield associated with image or virtual vorticity distributions (or, for that matter, source-sink distributions) outside R. In general, with specified velocity boundary condition v_b, the second integral in Eq. (6) can be evaluated explicitly for each point r of interest by numerical quadrature. For the important problem of a finite solid body executing time-dependent or steady translation and rotation in a fluid extending to infinity, this second integral can be reexpressed in terms of the instantaneous rectilinear velocity of the solid, $-v_\infty$, and the instantaneous angular velocity of the solid Ω. The resulting integral representation for the velocity vector is[21]

$$v(r,t) = -\frac{1}{A}\left[\int_R \frac{\omega(r_0,t) \times (r_0 - r)}{|r_0 - r|^d} dR_0 + 2\int_{R'} \frac{\Omega(r_0,t) \times (r_0 - r)}{|r_0 - r|^d} dR_0\right] + v_\infty(t) \quad (7)$$

where R' is the region occupied by the solid body and the reference frame is translating with the solid body at the velocity $-v_\infty$ relative to the freestream but not rotating with it.

The contribution of the vorticity distribution in R to the velocity field is given by the first integral in Eq. (6) or (7). The vorticity $\omega(r,t)$ originates from the solid boundary S, and at any finite time τ is nonnegligible only within a finite distance from S. The contributions of the vorticity to the velocity field therefore approach zero as r goes to infinity. Thus, the freestream velocity boundary condition is satisfied exactly at infinity. The use of the integral representation for the kinematics of the problem therefore eliminates the need of specifying a velocity boundary condition at a finite distance from the solid body. The difficulty of farfield boundary condition is thus removed.

Extraneous Boundary Condition

The kinetic processes of vorticity diffusion and convection in an incompressible fluid is described by Eq. (5) which is parabolic in its time-space relation. In order to solve Eq. (5) and obtain the time dependent vorticity field $\omega(r,t)$ in R, the values of ω on the boundary B must be known. The boundary B may contain one or more solid boundaries S. The "boundary values" of ω on S may not be computed by solving Eq. (5), since the local generation or depletion of vorticity on S is not described by the kinetic processes of vorticity transport.

To avoid the difficulties arising from the treatment of this extraneous boundary condition, several investigators suggested the use of p and v, rather than ω and v, as prime variables in the numerical procedure. It should be noted that for the incompressible flow the use of p and v substitutes the need to know p values on the boundary S for the need to know ω on S. The advantages offered by such a substitution, to the extent they exist, clearly are more than negated by the inability of the p and v formulation to confine the solution field to the viscous region.

Integral Law for Vorticity

The importance of satisfying integral laws for vorticity in the numerical solution of time-dependent viscous flow problems is recognized by many investigators. An integral law pertinent to the extraneous vorticity boundary condition is described in the following passage. Consider a flow past the exterior of a finite solid. The total vorticity Q in the fluid domain is given by

$$Q=\int_R \omega dR=-\int_S (\Omega\times r)\times n dB+\int_{S'} v_\infty \times n dB \qquad (8)$$

where S' is a surface at infinity, and the last expression is obtained in a non-rotating coordinate system using Gauss' theorem.

Since v_∞ is independent of the position vector, the second integral in Eq. (8) vanishes. Recognizing that S is the boundary of the solid domain, one can rewrite Eq. (8) upon the use of the theorem of Gauss as $Q=-2\Omega V$, where V is the volume of the solid. Since the vorticity of a solid rotating with an angular velocity Ω is 2Ω, the sum of the total vorticity of the fluid and of the solid is always zero. This "principle of total vorticity conservation" is valid for any arbitrary rotational and translational motion of a solid in a fluid which is at rest at infinity. It is obvious that Q does not change with time as long as the rotational motion of the solid does not undergo a change. That is, Q is conserved even if the translational motion of the solid changes with time. In particular, if a finite solid, initially at rest in a fluid also at rest, is set into a translational motion impulsively, Q is originally zero and remains to be zero after the solid is set into motion. The initial motion of the solid sets up, because of the no-slip condition on S, a vortex sheet on S. The total, or surface integrated, strength of this vortex sheet must be zero. As time progresses, the kinetical aspect of the problem prescribes the manner in which the vorticity spreads into the fluid. New vorticities are generated (or depleted) locally at the solid boundary S. The total vorticity in the fluid, however, does not change from zero.

generalized to fluids bounded internally and/or externally by any number of solids. This principle is based on kinematic considerations alone. It places certain restrictions on the vorticity boundary condition that are permissible in the numerical procedure. In particular, integrating Eq. (5) over R and using the theorem of Gauss, one obtains

$$\frac{\partial Q}{\partial t}=-\int_B (v\times\omega)\times n dB+\nu\int_B (\nabla\omega)\cdot\, n dB \qquad (9)$$

The boundary B consists of S and S'. The integrations over S' does not contribute to Eq. (9). For the first integral, the integration over S is reexpressible as a volume integral over R' and shown to be zero. Thus

$$\frac{\partial Q}{\partial t}=\nu\oint_S \frac{\partial\omega}{\partial n} dB \qquad (10)$$

As a consequence, the extraneous vorticity boundary condition must be determined in such a way that the integral in Eq. (10) gives the correct rate of change of Q. If the solid does not undergo a change in rotation, then the integral in Eq. (10) must b^ zero computationally.

One-Sided Difference Formulae

For simplicity, consider a two-dimensional flow in the x-y plane with S tangent to the x-axis. Denote a grid point at $x=0$, $y=mh$ by "m", and the velocity components by u, v. Then, using Taylor series at the origin for $u_{1/2}$, where the subscripts denote the grid points, one obtains, by noting that $\omega_0=-(\partial u/\partial y)0$ and $u_0=0$:

$$\omega_0=-\frac{2u_{1/2}}{h}+\left(\frac{\partial^2 u}{\partial y^2}\right)_0\frac{h}{4}+0(h^2)+\ldots \qquad (11)$$

If terms of order h and higher are neglected in Eq.(11) then one obtains a first-order one-sided formula for ω_0. The value of u at the grid point "½" is utilized here so that the formulae are reducible to those based on the stream function ψ which are often used in the literature. For example, by noting that $u_{1/2}=(\psi_1-\psi_0)/h+0\ (h^2)+\ldots$, one obtains from Eq., (11) a first order formula:

$$\omega_0=-2(\psi_1-\psi_\circ)/h^2 \qquad (12)$$

The formula (12) has been used by many previous investigators and is considered to be "safer" than second-order formulae. However, in a time-dependent computation, error introduced at the boundary propagates into the fluid domain as the solution progresses with time. It therefore appears that, if a first-order formula is used to estimate ω_0 values, there is little to be gained by using formulae accurate to an order higher than the first, for the computation of vorticity values away from S. An even more serious objection is that the first-order formula does not permit the permit the pressure gradient along S to be accounted for correctly. On S,since $v=0$, the x-component of the Navier-Stokes equation gives

$$-\left(\frac{\partial\omega}{\partial y}\right)_0=\left(\frac{\partial^2 u}{\partial y^2}\right)_0=\frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)_0 \qquad (13)$$

Note that in obtaining the first-order formula, the first term dropped from Eq. (11) is $(\partial^2u/\partial y^2)_0 h/4$. This is equivalent to neglecting the tangential pressure gradient effects in the estimate of ω_0. Alternatively, the dropping of the term may be viewed as specifying $(\partial\omega/\partial y)_0=0$ and thus, in effect, imposing a "Neumann" extraneous vorticity boundary condition on S. The imposition of this extraneous vorticity boundary condition fortuitously ensures the satisfaction of the principle of conservation of total vorticity, since the condition $\partial\omega/\partial n=0$ on S clearly rendered the integral in Eq. (10) zero. In general, however, the first-order formula does not describe the local generation of vorticity on S adequately.

The inadequacy of the first-order formulae can be demonstrated by examining the problem of flow formation in Couette motion with a pressure gradient. Consider the fluid between two parallel infinite plates separated by the distance L. The lower plate coincides with the $y=0$ plane. Initially the fluid and both plates are at rest. Starting from the time $t=0$, the upper plate is set into a constant motion in the x-direction with a velocity U, and a constant pressure gradient dp/dx is maintained in the fluid.

The differential equation of motion for this problem in terms of u contains a pressure gradient term, and is well known. In the vorticity transport equation, however, the pressure gradient term does not appear explicitly. Thus the dependence of ω on the pressure gradient can enter the problem only through the boundary and initial conditions for ω. It is obvious that the initial distribution of vorticity at $t=0^+$ is independent of the pressure gradient. If a first-order one-sided formula, which implies the conditions $(\partial\omega/\partial y)_{y=0}=0$ and $(\partial\omega/\partial y)_{y=L}=0$, is used to determine subsequent vorticity distributions, then, since both the initial and the boundary conditions for ω are independent of the pressure gradient, these subsequent vorticity distributions will all be independent of the pressure gradient. In fact, it has been shown computationally that the asymptotic steady solution gives

$\omega = U/L$, a constant. This steady solution as well as the transient solution obtained, correspond to the special case of zero pressure gradient.

The second-order formulae, which takes into account the tangential pressure gradient effect, can violate the principle of total vorticity conservation. To illustrate this point, consider the Rayleigh's problem for an infinite flat plate. This plate coincides with the $y=0$ plane, and is set into motion with the velocity U at $t=0$, and thereafter maintains this velocity. The vorticity transport equation for this problem expressed in an explicit finite-difference form is

$$\omega^{n+1}_{m+\frac{1}{2}} = \omega^{n}_{m+\frac{1}{2}} + \frac{\nu \Delta t}{h^2}\left(\omega^{n}_{m+3/2} - 2\omega^{n}_{m+\frac{1}{2}} + \omega^{n}_{m-\frac{1}{2}}\right) \quad (14)$$

where the superscripts refer to time, and "midpoint" vorticity values are used to enable the evaluation of the total vorticity by a simple summation procedure. From Eq. (14), one obtains by summing each term from $m=0$ to $m=\infty$, and neglecting terms of order h^3 or higher,

$$Q^{n+1} - Q^{n} = \left(\frac{\nu \Delta t}{h^2}\right) h \left(\omega^{n}_{-\frac{1}{2}} - \omega^{n}_{\frac{1}{2}}\right) \quad (15)$$

The point "$-\frac{1}{2}$" is outside the fluid domain and is introduced for convenience. If a second-order formula is used to estimate the value at this point, one has $\omega_{-\frac{1}{2}} = 3\omega_{\frac{1}{2}} - 2\omega_{3/2}$. Consequently, the right side of Eq. (15) becomes $2h(\nu\Delta t/h^2)(\omega^{n}_{\frac{1}{2}} - \omega^{n}_{3/2})$. Since in general $\omega_{3/2} \neq \omega_{\frac{1}{2}}$, the right side of Eq. (15) does not vanish, and the principle of total vorticity conservation is not satisfied. The seriousness of the computational error introduced into Q can be assessed with the aid of the known analytical solution for this problem. Let $t = n\Delta t$ and $\nu\Delta t/h^2 = \frac{1}{2}$, the maximum value permitted by the numerical stability criterion. Evaluating $\omega^{n}_{\frac{1}{2}}$ and $\omega^{n}_{3/2}$ using the exact analytical solution, and placing the results in Eq. (15), one has

$$Q^{n+1} - Q^{n} = U\left(\frac{2}{\pi n}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{8n}\right)\left[1 - \exp\left(-\frac{1}{n}\right)\right] \quad (16)$$

The correct value for Q is U. The above formula indicates that if the solution contains no error at the tenth time level, then a 2.4% error is introduced into the value of Q at the eleventh time level. The accumulation of errors of this magnitude clearly leads to unacceptable results after a limited number of time steps. For the present one-dimensional problem, the error in Q is reflected in a change of the freestream velocity as the computation progresses. For two- and three-dimensional problems, errors introduced into Q at various grid points may have opposite signs, and the sum of these errors introduced at each time step may be substantially smaller than that for the one-dimensional problem. At high Reynolds numbers, however, a very large number of time steps is often requrued to obtain a near steady-state solution so that the accumulated error may be large.

It should be noted that if a first-order formula is used, then $\omega_{-\frac{1}{2}} = \omega_{\frac{1}{2}}$ and the right side of Eq. (15) becomes zero. The first term neglected is $0(h^3)$. Therefore the first-order formulae introduce smaller error into Q than the second-order formulae. This fact is consistent with the observation by many investigators that the first-order formulae tend to be more stable than the second-order ones.

The above analyses do not lead to the conclusion that all second-order formulae are unstable. Indeed the literature contains several articles in which second-order formula were used, and stable solutions were obtained at relatively high Reynolds numbers. These formulae, however, are usually ad-hoc, and the above analyses re-enforce the observation that there are serious uncertainties regarding the range of usefulness of each second-order formula, and the reasons for the difficulties experienced. The analyses above suggests that the violation of the integral law for vorticity is a plausible reason for these difficulties.

Kinematic Treatment

The present method for treating the vorticity boundary condition on S is based on the kinematic relationship between the instantaneous velocity and vorticity distributions. The stress-strain relations, which differentiates the kinetic behavior of a solid from that of a fluid, does not enter the kinematic relationship. There is therefore no need to exclude the solid domain from the field of interest in the kinematics of the problem. If at any instant of time the vorticity distribution is known in the unlimited space, including both the solid region R' and the fluid region R, then the corresponding velocity distribution is determinate through the Biot-Savart Law in both R' and R. Indeed, Eq. (7) may be obtained by noting that the instantaneous vorticity in R' is simply 2Ω. This equation is valid in R' as well as in R.

Consider the problem involving prescribed solid body motions. The instantaneous vorticity distribution throughout the unlimited space $R+R'$ must be such that the velocity in the solid region R' as computed from Eq. (7) agrees with the known solid motion. In other words, the prescribed solid motion imposes a kinematic restriction on the vorticity distribution. Since the vorticity in the solid region R' is known to be 2Ω, a specified value, the kinematic restriction is imposed only on the vorticity distribution in the fluid. For the present discussion, let S be composed of two adjacent boundaries S^+ and S^-, separated by infinitesimal, distances from one another, S^+ being a part of R, and S^- a part of R'. At the time level $\tau + \Delta t$, the new free vorticity distribution in the fluid away from S^+ is computed from the kinetic equation of vorticity transport. Therefore only the new vorticity distribution on S^+ is subject to the kinematic restrictions. This surface vorticity distribution, designated ζ, represents a vortex sheet on S^+, and has the dimension of velocity. In the numerical procedure, new free vorticity values are computed for data points at finite distances from S. New ζ values are then computed at surface data points in accordance with the kinematic restriction. Surface vorticity values are then determined by considering the vorticity represented by ζ to be uniformly spread over a normal distance which is one half the distance between S and an adjoining data surface.

Consider the two-dimensional exterior flow problem involving a single nonrotating finite solid. With $\Omega = 0$, the velocity field in R' is irrotational and solenoidal. Thus a potential function φ exists in R' such that $v = \nabla\varphi$, and $\nabla^2\varphi = 0$. Let b be a unit tangent vectors on S^-, and b, n, ζ/ζ be a right-handed set of orthogonal unit vectors. Let v_b and v_n be the tangential and normal velocity components. If ζ is found such that $v_b = \partial\varphi/\partial b = 0$ on S^-, then φ is a constant on S^-. The principle of maximum then requires φ to be a constant throughout R'. Therefore $v=0$ in R'. Also, the divergence theorem gives $\int_{R'} v \cdot v dR = \int_{S^-} \phi v_b dB$. If ζ is found such that $v_n = 0$ on S^-, then clearly $v=0$ in R', provided that the size of R' is not zero. As a consequence, in the computation of ζ, either $v_b = 0$ on S^- or $v_n = 0$ on S^-, and not both, needs to be required. For the special case where the size of R' is zero (plates of infinitesimal thickness), the requirement $v_n = 0$ on S^- does not ensure $v=0$ in R'. Therefore the proper requirement for this special case is $v_b = 0$ on S^-.

The requirement $v_b = 0$ on S^- gives, using Eq.(7),

$$\frac{1}{2\pi}\int_{S^+} \frac{\zeta_0 (r_{on} - r_n)}{|r_0 - r|^2} dB_0 = -(F + v_\infty)_b \text{ on } S^- \quad (17)$$

where

$$F = -\frac{1}{2\pi}\int_{R-S^+} \frac{\omega_0 \times (r_0 - r)}{|r_0 - r|^2} dR_0 \quad (18)$$

Since the right side of Eq. (17) is known at the time level $\tau+\Delta t$, Eq. (17) is a Fredholm integral equation containing ζ at the time level $\tau+\Delta t$ as the unknown function. The theory of solution of Eq. (17) is discussed extensively in a number of treatises.[22,23] Numerical methods of solution for integral equations corresponding exactly to Eq. (17) have been developed by A.M.O. Smith and his co-workers[24,25] for computing potential flows about solids. The implementation of the present method for calculating ζ values on S^+ is straightforward, since accurate and well-tested techniques developed for the potential flow calculations can be used directly here.

It should be pointed out that with Lighthill's approach, one constructs a potential velocity field v_p in R such that $(v_p+F+v_\infty)_n=0$ on S^-. In general, a nonzero v_p in R implies a nonzero virtual or image vorticity in R'. Consequently, the discussion immediately prior to Eq. (17) is not useful with Lighthill's approach. In other words, one finds $(v_p+F+v_\infty)_b \neq 0$ in general. In fact, ζ is determined from this nonzero quantity in Lighthill's approach.

The numerical procedure developed by Smith et al. employs source-sink singularities distributed over S^-. This singularity distribution is determined by requiring the normal velocity to satisfy the prescribed boundary condition on S^+. The present method deals with vorticity singularities distributed over S^+. The prescribed tangential velocity condition is to be satisfied on S^-. The differences in the types of singularities used in the methods and in the types of boundary conditions are unimportant since the integral equations treated are identical except for the nomenclature. The reversal of the role of S^- and S^+ makes the present exterior problem, in so far as the computation of the singularity distribution is concerned, equivalent to an interior potential flow problem. As a consequence, the present solution of Eq. (17) for the exterior flow problem exists only under certain essential conditions, and is not unique when it exists.[22] It can be shown that the essential condition for the existence of a solution for the vorticity distribution on S^+ is that the integral over S^- of $F_b+v_{\infty b}$ is zero.[22] This condition is clearly met since the vorticity distribution is non-zero only exterior of S^-.

The nonuniqueness of the solution is evidenced by the fact that there are distributions of singularities on S^+, not identically zero, that give zero velocity everwhere interior of S^+ (and hence also on S^-). Such distributions shall be denoted by ζ'. For example, ζ' = constant on a circular boundary S^+ is one such distribution. It should be noted that for the interior potential flow problem as treated by Smith et al. one is not concerned with the flow outside R'. The nonuniqueness of the singularity distribution is immaterial since ζ' do not alter the velocity field in the region of interest. For the present exterior flow problem, however, S^+ is in the flow region of concern. The velocity field in R may be indeterminate if the distribution of vorticity on S^+ is not uniquely known. The constraint imposed by the principle of total vorticity conservation, however, resolves this difficulty.[21] This principle requires that

$$\oint_S (\zeta+C\zeta')\,dB+\int_R \omega\,dR=0 \tag{19}$$

where ζ is any distribution that satisfies Eq. (17), and C is a constant whose magnitude is to be determined. In Ref. 21, it is shown that for the two-dimensional exterior flow problem, a unique solution is obtained from Eqs. (17) and (19). The analyses for the three-dimensional and interior flow problems are similar to those give there for the two-dimensional exterior problem. In fact, they are slightly less involved, since for the two-dimensional exterior problems the flowfield R is doubly connected, and for the other problems the flowfield of concern R is simply connected.

Results and Discussions

Rayleigh's impulsively started infinite plate problem was solved using an explicit finite-difference equation for the vorticity transport equation. Various methods for computing the vorticity values on the contact surfaces were used. It was found that the first-order one-sided difference formula gave numerical results in excellent agreement with the exact solution. The second-order one-sided difference formula led to unstable results. The new method for determining the vorticity values gave results in excellent agreement with the exact solution.

The agreement and disagreement with the exact solution found for the one-dimensional Rayleigh's problem was exactly as expected from the earlier discussions. The crucial test of the new method, however, is its application to multidimensional problems. Two two-dimensional problems have been studied thus far, that of a finite plate set into motion impulsively in its own plane in a direction perpendicular to its leading edge, and that of a circular cylinder set into motion impulsively in a direction perpendicular to its axis. In both problems, the symmetry of the flow ensures the conservation of the total vorticity. For both problems, the velocity field was computed by a numerical quadrature of the integral representation, Eq. (7). The time-dependent vorticity transport equation was treated, and steady state solutions obtained in the limit of large time.

Numerical results were obtained in the present study for the circular cylinder problem at a Reynolds number based on the cylinder diameter of 40. A simple, forward-time and centered-space, explicit finite-difference method was used for the vorticity transport equation. The steady-state pressure distribution and drag coefficients obtained are in reasonable agreement with experimental data,[27,28] as shown in Figs. 1 and 2. Also shown is the computed drag coefficient for a Reynolds number of 1000, obtained using a DuFort-Frankel method for vorticity transport. In view of the fact that vortex shedding and possible wake turbulence are not considered in the present study, the good agreement between the computed drag and the experimental value at the Reynolds number of 1000, shown in Fig. 2, may be fortuitous. It is noted, however, that the numerical results of Thoman and Szewczyk[29] considering vortex shedding showed only slight oscillations in the drag values. Also, at a Reynolds number of 1000, the location of transition to turbulence may be far from the cylinder so that the flow near the cylinder is not substantially influenced by wake turbulence. The computer time used was 14 minutes on the UNIVAC 1108 for the Reynolds number of 40 and 30 min. for the Reynolds number of 1000.

The circular cylinder problem is a popular test problem, and has been treated by many previous investigators. Reference 29 contains comparisons of many of the previous results. The present drag results agree with experimental data better than the earlier results[7] obtained using the integro-differential method and a one-sided difference formula for the boundary vorticity computation, as shown in Fig. 2. This better agreement, however, should not be attributed entirely to the present method of treating the extraneous vorticity boundary condition, since the earlier results were obtained using a rectangular grid work, while the present results were obtained using a more suitable polar grid work.

Of greater significance to the present study are the results for the flat plate problem which was selected for extensive study because of its relatively simple geometry and because detailed velocity fields based on the boundary-layer theory, on finite-difference method, and on Oseen's approach are available for comparison with present results. Numerical results were obtained for the case of a Reynolds number based on the plate length L of 4 using the simple explicit finite-difference equation for the computation of vorticity.

The tangential velocity profile, u, at midplate obtained using the present method for computing the vorticity values on the plate was compared with the "best run" finite-difference profile,[30] and Oseen's profile[31] in Fig. 3. Near the plate, the 3 sets of results were found to be in good agreement. Far from the plate, where Oseen's results are not available,

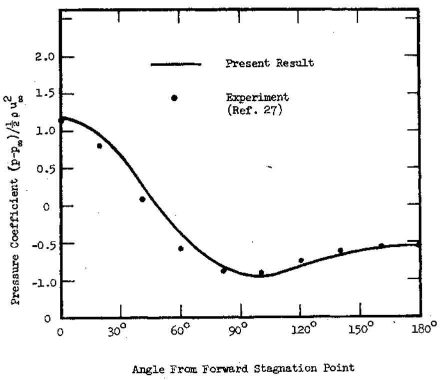

Fig.1 Pressure distribution over a circular cylinder (Reynolds number 40).

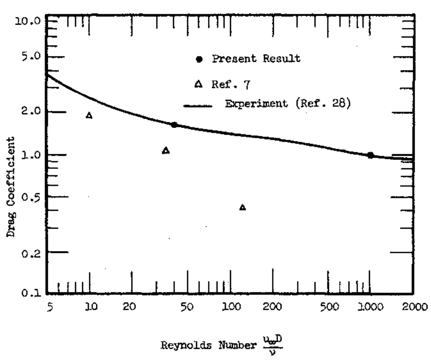

Fig.2 Drag coefficient for circular cylinder.

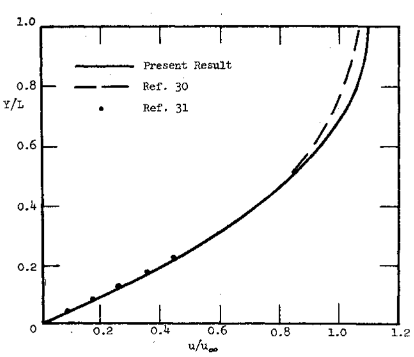

Fig. 3 Mid-plate velocity profiles (Reynolds number 4).

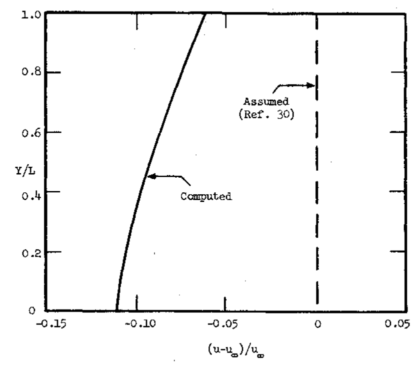

Fig. 4 Farfield velocity condition one plate length upstream (Reynolds number 4).

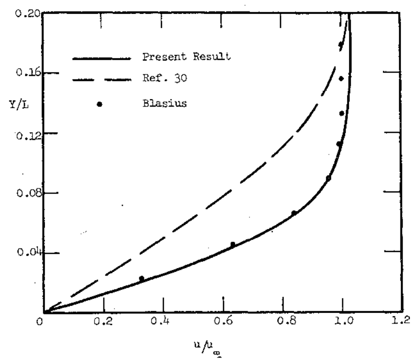

Fig.5 Mid-plate velocity profiles (Reynolds numbers 1000 and 993).

the present result shows a more pronounced velocity overshoot ($u > u_\infty$) than the finite-difference result. The overshoot is expected, and is attributed to the favorable pressure gradient along the plate caused by displacement thickness. The deviation of the finite-difference result from the present result is also expected because of the difference in the specification of "far field" velocity boundary condition. In Ref. 30, u is set to be u_∞ on a boundary one plate length upstream of the leading edge. The value of u at this upstream boundary was computed by the integral-representation of v, and shown in Fig. 4. This figure shows that at this upstream boundary, the value of u differs significantly from the freestream value. The results of Ref. 30 was obtained using an ADI method for the vorticity computation. The computer time used was 90 min on a UNIVAC 1108 computer. The authors of Ref. 30 estimated that their ADI method was two to three times faster than the explicit method which they also attempted. Therefore, the explicit method of Ref. 30 would require 180 to 270 min. for the steady-state solution. In comparison, the present results were obtained with the explicit method for the vorticity computation using a UNIVAC 1108 time of 40 min. A flowfield segmentation technique, presented in Ref. 8 and based on the integro-differential approach, further reduced the computer time used to 14 min.

Additional results have been obtained for the flat plate problem at a Reynolds number of 1000 using an ADI method for the vorticity transport equation. The steady-state mid-plate velocity profile obtained are shown in Fig. 5. As expected, the present result agrees well with the Blasius' profile for a semi-infinite plate at a station half the finite-plate length from the leading edge. The velocity overshoot is still present, but is smaller in magnitude than that for the Reynolds number 4 case. This behavior is consistent with the fact that the displacement thickness decreases with increasing Reynolds number. For the case with a relatively high Reynolds number of 993, the farfield boundary condition used in Ref. 30 is actually quite accurate. The mid-plate velocity profile presented in Ref. 30 for the Reynolds number 993 case, however, deviated greatly from the corresponding Blasius' profile, as shown in Fig. 5. The drag coefficient computed in Ref. 30 is 0.01846, that computed here is 0.04629, while the experimental data is 0.0460. It is noted that a one-sided difference formula was used to estimate the boundary vorticity values whereas the present, apparently more accurate, results were obtained using the kinematic treatment of the vorticity boundary conditions. In Ref. 30, a coordinate transformation was used to minimize the number of data points needed, resulting in a UNIVAC 1108 time of 60 min. The present results were obtained using uniform grid spacing, and 12 min. of UNIVAC 1108 time.

Based on the results of the present work, it is concluded that the present method for treating the farfield and extraneous boundary conditions simulates the physical problem numerically in an accurate manner. Previous difficulties and uncertainties relating to the numerical boundary conditions are resolved for time-dependent incompressible flows.

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